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Generalized Jacobi Chebyshev Wavelet Approximation: Perkiraan Wavelet Jacobi Chebyshev Umum

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General Background: Wavelet approximations are fundamental in numerical analysis and signal processing, with classical orthogonal polynomials like Jacobi and Chebyshev serving as key tools due to their strong approximation properties. **Specific Background:** The use of Chebyshev wavelets has been extended through generalized polynomial frameworks, such as Koornwinder's generalization of Jacobi polynomials, offering more flexibility for function approximation on finite intervals. **Knowledge Gap:** Despite existing wavelet frameworks, the integration of generalized Jacobi and Chebyshev structures into a unified wavelet approximation scheme remains underexplored. **Aims:** This study introduces the Generalized Jacobi Chebyshev Wavelet (GJCW) approximation, establishing its theoretical foundations and demonstrating convergence and approximation capabilities. **Results:** It is shown that for a uniformly bounded function expanded in the GJCW basis, the partial sums yield both convergent and best uniform polynomial approximations. **Novelty:** The formulation of a new wavelet approximation based on a hybrid of generalized Jacobi and Chebyshev polynomials constitutes a novel contribution, supported by rigorous recurrence relations and multiresolution analysis. **Implications:** This work enhances the theoretical landscape of wavelet-based function approximation, with potential applications in computational mathematics, signal analysis, and numerical solutions of differential equations.

Highlight :

- Wavelet Construction: The paper defines and constructs *generalized Jacobi Chebyshev wavelets* using orthogonal polynomials.
- Approximation Theory: It proves that if the wavelet series converges, then a *uniform best polynomial approximation* exists.
- Multiresolution Framework: The approach is grounded in *Mallat's multiresolution analysis*, enabling efficient function approximation.

Keywords : Jacobi Polynomials, Chebyshev Wavelets, Multiresolution Analysis, Polynomial Approximation, Orthonormal Basis

INTRODUCTION

Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ constitute a category of classical orthogonal polynomials. They are orthogonal about the weight $(1-x)^\alpha (1+x)^\beta$ on the interval [-1,1]. . We have $P_n^{((\alpha,\beta))}(x)$ Chebyshev polynomials with $|x|\leq 1$.

n	*Jacobi Polynomial*
0	1
1	$(\alpha+\beta)+\frac{1}{2}(\alpha+\beta+2)x$
2	$-4+\alpha^2-\beta+\beta^2-\alpha(1+2\beta)+2(3\alpha+\alpha^2-\beta(3+\beta))x$
|	$+\frac{1}{8}(3+\alpha+\beta)(4+\alpha+\beta)x^2$
3	$(\alpha+\beta)(-16+\alpha^2+(-3+\beta)\beta-\alpha(3+2\beta))+3(\alpha+\beta+4)$
	$(\alpha^2-(2\beta+1)\alpha+\beta^2-\beta-6)x+3(\alpha+\beta(\alpha+\beta+4)(\alpha+\beta+5)x^2$
	$+\frac{1}{48}((\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)x^3$

Figure 1.

also for n=4,5,

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\}.$$

Figure 2.

The Jacobi polynomials are generated by the three-term recurrence relation, for scalars a_n^((α,β)),b_n^((α,β)) and , c_n^((α,β)),

$$P_{n+1}^{(\alpha,\beta)}(x)=(a_n^{(\alpha,\beta)}x-b_n^{(\alpha,\beta)})P_n^{(\alpha,\beta)}(x)-c_n^{(\alpha,\beta)}P_{n-1}^{(\alpha,\beta)}(x)\ \ n\geq 1,$$

Figure 3.

where

$$P_0^{(\alpha,\beta)}(x)=1,\ P_1^{(\alpha,\beta)}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta),$$

Figure 4.

also

$$a_n^{(\alpha,\beta)} = \frac{(2m+\alpha+\beta+1)(2m+\alpha+\beta)}{2(m+1)(m+\alpha+\beta+1)},$$
$$b_n^{(\alpha,\beta)} = \frac{(\beta^2-\alpha^2)(2m+\alpha+\beta+1)}{2(m+1)(m+\alpha+\beta+1)(m+\alpha+\beta)},$$
$$c_n^{(\alpha,\beta)} = \frac{(2m+\alpha+\beta+2)(m+\alpha)(m+\beta)}{(m+1)(m+\alpha+\beta+1)(2m+\alpha+\beta)}.$$

Figure 5.

The Jacobi polynomials y=P_n^((α,β)) (x) solve the linear second-order differential equation

$$(1-x^2)y'' + [\beta - \alpha - (\alpha+\beta+2)x]y' + n(n+\alpha+\beta+1)y = 0.$$

Figure 6.

A. Jacobi polynomials

Theorem 1.1 *There are scalars* a_0^((α,β)),a_1^((α,β)),…,a_n^((α,β))∈R *such that*

P_n^((α,β)) (x)=∑_(m=0)^n▒ a_(n,m)^((α,β)) x^m.

Proof. We use mathematical induction.

For this

P_0^((α,β)) (x)=1,

P_1^((α,β)) (x)=1/2(α+β+2)x+1/2(α-β).

The hypothesis of mathematical induction:

Suppose for 0≤k<n , we have

$$P_k^{(\alpha,\beta)}(x) = \sum_{m=0}^{k} c_{n,m}^{(\alpha,\beta)} x^m,$$

Therefore

$$P_{n-1}^{(\alpha,\beta)}(x) = \sum_{m=0}^{n-1} c'^{(\alpha,\beta)}_{n,m} x^m$$

and

$$P_{n-2}^{(\alpha,\beta)}(x) = \sum_{m=0}^{n-2} c''^{(\alpha,\beta)}_{n,m} x^m,$$

Figure 7.

The rule of mathematical induction:

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For scalars a_n^((α,β)),b_n^((α,β)) and c_n^((α,β))

$$
\begin{aligned}
P_n^{(\alpha,\beta)}(x) &= (d_n^{(\alpha,\beta)}x - e_n^{(\alpha,\beta)})P_{n-1}^{(\alpha,\beta)}(x) + f_n^{(\alpha,\beta)}P_{n-2}^{(\alpha,\beta)}(x) \\
&= (c_0^{(\alpha,\beta)}c'^{(\alpha,\beta)}_{n,0} - f_0^{(\alpha,\beta)}c''^{(\alpha,\beta)}_{n,0}) + \sum_{m=1}^{n-2}[c'_{n,m-1} - c_n^{(\alpha,\beta)}c_{n,m}^{\alpha,\beta)} - f_n^{(\alpha,\beta)}c''_{n,m}]x^m \\
&\quad +(c_{n-2}^{(\alpha,\beta)}c'^{(\alpha,\beta)}_{n,n-2} - f_{n-2}^{(\alpha,\beta)}c''^{(\alpha,\beta)}_{n,n-2})x^{n-2} + c_{n-1}^{(\alpha,\beta)}c'^{(\alpha,\beta)}_{n,n-1}x^{n-1} + c'^{(\alpha,\beta)}_{n,n-1}d_n^{(\alpha,\beta)}x^n \\
&= \sum_{m=0}^{n} h_{n,m}^{(\alpha,\beta)}x^m.
\end{aligned}
$$

Figure 8.

Theorem 1.2

Let $P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} a_{n,m}^{(\alpha,\beta)}x^m$ be a representation of Jacobi polynomials. Then The following conditions are satisfying:

i) $a_{n,m}^{\alpha,\beta} = 0$ for all $m \geq n$,

ii) $(m-1)(m+2)a_{n,m+2}^{\alpha,\beta} - (\beta-\alpha)(m+1)a_{n,m+1}^{\alpha,\beta} + [-m(m-1) - m(\alpha+\beta+2) + n(n+\alpha+\beta+1)]a_{n,m}^{\alpha,\beta} = 0$ for all $0 \leq m \leq n$.

Proof. Suppose $y = P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} a_{n,m}^{(\alpha,\beta)}x^m$, then

$$y' = \sum_{m=1}^{n} m a_{n,m}^{(\alpha,\beta)}x^{m-1} = \sum_{m=0}^{n-1}(m+1)a_{n,m+1}^{(\alpha,\beta)}x^m$$

$$y'' = \sum_{m=2}^{n} m(m-1)a_{n,m}^{(\alpha,\beta)}x^{m-2} = \sum_{m=0}^{n-2}(m+1)(m+2)a_{n,m+2}^{(\alpha,\beta)}x^m.$$

We put in above equation we have

$$
\begin{aligned}
&\sum_{m=0}^{n-2}(m+1)(m+2)a_{n,m+2}^{(\alpha,\beta)}x^m - \sum_{m=2}^{n} m(m-1)a_{n,m}^{(\alpha,\beta)}x^m + (\beta-\alpha)\sum_{m=0}^{n-1}(m \\
&\quad + 1)a_{n,m+1}^{\alpha,\beta}x^m - ((\alpha+\beta+2)\sum_{m=0}^{n} m a_{n,m}^{\alpha,\beta}x^m + n(n+\alpha+\beta \\
&\quad + 1)\sum_{m=0}^{n} a_{n,m}^{\alpha,\beta}x^m = 0,
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&\sum_{m=0}^{n} -(m-1)(m+2)a_{n,m+2}^{\alpha,\beta} - (\beta-\alpha)(m+1)a_{n,m+1}^{\alpha,\beta} \\
&+[-m(m-1) - m(\alpha+\beta+2) + n(n+\alpha+\beta+1)]a_{n,m}^{\alpha,\beta}x^m = 0.
\end{aligned}
$$

Figure 9.

B. Generalized Jacobi Chebyshev Wavelets

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Let $\alpha > -1, \beta > -1, M \geq 0$ and $N \geq 0$. Koornwinder in [12] it is demonstrated that the generalised Jacobi polynomials $\{P_n^{(\alpha,\beta,M,N)}(x)\}_{n=0}^{\infty}$ can be written as

$$P_n^{(\alpha,\beta,M,N)}(x) = P_n^{(\alpha,\beta)}(x) + MQ_n^{(\alpha,\beta)}(x) + NR_n^{(\alpha,\beta)}(x) + MNS_n^{(\alpha,\beta)}(x), \ n = 0,1,2, \ldots,$$

where

$$Q_0^{(\alpha,\beta)}(x) = R_0^{(\alpha,\beta)}(x) = S_0^{(\alpha,\beta)}(x) = 0,$$

and for $n = 1,2,3, \cdots$,

$$Q_n^{(\alpha,\beta)}(x) = \frac{(\beta+2)_{n-1}(\alpha+\beta+2)_{n-1}}{(\alpha+1)_n n!} \times [n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) - (\beta+1)(x-1)DP_n^{(\alpha,\beta)}(x)],$$

$$R_n^{(\alpha,\beta)}(x) = \frac{(\alpha+2)_{n-1}(\alpha+\beta+2)_{n-1}}{(\beta+1)_n n!} \times [n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) - (\alpha+1)(x+1)DP_n^{(\alpha,\beta)}(x)],$$

and

$$S_n^{(\alpha,\beta)} = \frac{(\alpha+\beta+2)_n(\alpha+\beta+2)_{n-1}}{(\alpha+1)(\beta+1)n!(n-1)!} \times [n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) - \{(\beta+1)(x-1) + (\alpha+1)(x+1)\}DP_n^{(\alpha,\beta)}(x).$$

Since $P_n^{(\alpha,\beta,M,N)}(x), R_n^{(\alpha,\beta)}, Q_n^{(\alpha,\beta)}, S_n^{(\alpha,\beta)}$ are linear combination of $P_n^{(\alpha,\beta)}, DP_n^{(\alpha,\beta)}$, we have

Figure 10.

Proposition 2.1 *The generalized Jacobi polynomials* $P_n^{\alpha,\beta,M,N}(x)$ *are generated by the three-term recurrence relation, for scalars* $d_n^{(\alpha,\beta,M,N)}, e_n^{(\alpha,\beta,M,N)}$ *and* $f_n^{(\alpha,\beta,M,N)}$, *for* $n \geq 1$

$$P_{n+1}^{(\alpha,\beta,M,N)}(x) = (d_n^{(\alpha,\beta,M,N)}x - e_n^{(\alpha,\beta,M,N)})P_n^{(\alpha,\beta,M,N)}(x) - f_n^{(\alpha,\beta,M,N)}P_{n-1}^{(\alpha,\beta,M,N)}(x).$$

In the following table, we define generalized Chebyshev wavelets. (see [2, 4, 7, 8, 9, 10,11,15])

Suppose $k \in N$ (degree of multiresolution), $m \geq 0, n = 1,2, \cdots, 2^k$

$$\Psi_{n,m}^{(\alpha,\beta,M,N)}(t) = \begin{cases} \sqrt{\frac{2^{k+1}}{n}} P_m^{(\alpha,\beta,M,N)}(2^k t - 2n + 1)) & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}], \\ 0 & other9wise \end{cases}$$

A function $f \in L^2[-1,1)$ is expanded by generalized Chebyshev wavelets series as

$$f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} c_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t),$$

where

$$c_{n,m} = \int_{-1}^{1} f(t)\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)\omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt,$$

and $\omega_{n,m}^{(\alpha,\beta,M,N)}$ is the weight function of (α, β, M, N) generaliized Chebyshev polynomials. Also

$$\int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt = L_{n,m}^{(\alpha,\beta,M,N)},$$

Figure 11.

It is necessary to study multiresolution analysis and Mallat’s Theorem for wavelet approximation.

Definition 2.1 Multiresolution Analysis: *An MRA with scaling function ϕ constitutes a collection of closed subspaces*

$\{V_j\}_{j\in Z}$ *of* $L^2(\mathbb{R})$, *such that*

(i) $V_j \subset V_{j+1}$;

(ii) $f(x) \in V_j \Leftarrow f(2x) \in V_{j+1}$;

(iii) $\overline{\cup V_j} = L^2(\mathbb{R})$,

(iv) $\cap V_j = 0$;

(v) There exists a function $\phi \in V_0$ such that the collection $\{\phi(x-k): k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

Figure 12.

The series of wavelet subspaces W_j of $L^2(\mathbb{R})$ is such that $V_j \perp W_j$, for all j and $V_{j+1} = V_j \oplus W_j$. Closure of $\oplus W_j$ is dense in $L^2(\mathbb{R})$ for L^2 norm.

Figure 13.

We now present Mallat’s theorem, which ensures that in the context of an orthogonal multiresolution analysis (MRA), an orthonormal basis exists for $L^2(R)$ exists. These basis functions are essential in wavelet theory, facilitating the development of sophisticated computational algorithms.

Lemma 2.1 (Mallat's Theorem) In the context of an orthogonal multiresolution analysis (MRA) characterised by a scaling function ϕ, a corresponding wavelet exists $\psi \in L^2(R)$ *such that for each* $j \in Z$ *, the family* $\{\psi_{j,k}\}_{k\in Z}$ *is an orthonormal basis for* W_j *. Hence the family* $\{\psi_{j,k}\}_{k\in Z}$ *is an orthonormal basis for* $L^2(R)$ *.*

Definition 2.2 *(i) Let family be an orthonormal basis for* $L^2(R)$ *and* $P_n(f)$ *the orthogonal projection of* $L^2([-1,1])$ *onto* V_n *. Then*

$$P_n(f)=\sum_{-\infty}^{\infty} \langle f,\psi_{n,k}\rangle \psi_{n,k}, n=1,2,3,\ldots$$

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(ii) The wavelet approximation of the Chebyshev polynomial is defined by

$$E_n(f) = \| f - P_n(f) \|_2 = \int_{-1}^{1} |f(t) - P_n(f)(t)|^2 dt =$$

$o(\phi(n)),\quad o(\phi(n))\ is\ a\ small\ function.$

Theorem 2.1 *Let* $f \in L^2([-1,1])$ *be a uniformly bounded function and* $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)}(t)$ be expanded in terms of generalized Chebyshev wavelets and the series $\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ be convergent. Then generalized Chebyshev wavelet approximation f, for every M is the partial sums

$$u_{2^k,M-1}^{(\alpha,\beta,M,N)}(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{M-1} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)}(t),$$

and

$$E_{2^k,M-1}(f) = o\Big(\big(\sum_{n=0}^{2^k} \sum_{m=M}^{\infty} L_{n,m}^{(\alpha,\beta,M,N)} |t_{n,m}|^2\big)^{\frac{1}{2}}\Big).$$

Figure 14.

Proof. We have

$$\begin{aligned}
&\| f - u_{2^k,M-1^{(\alpha,\beta,M,N)}} \|_2^2 \\
&= \int_{-1}^{1} \Big| \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)}(t) \\
&- \sum_{n=0}^{2^k} \sum_{m=0}^{M-1} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)}(t) \Big|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)\, dt \\
&= \int_{-1}^{1} \Big| \sum_{n=0}^{2^k} \sum_{m=M}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)}(t) \Big|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)\, dt \\
&\le \sum_{n=0}^{2^k} \sum_{m=M}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)\, dt \\
&= \sum_{n=0}^{2^k} \sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}
\end{aligned}$$

Therefore $\| f - s_{M-1} \|_2 \le \big(\sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}\big)^{\frac{1}{2}}$. That is

$$E_{2^k,M-1}(f) = o\Big(\big(\sum_{n=0}^{2^k} \sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}\big)^{\frac{1}{2}}\Big),$$

Figure 15.

Definition 2.3 *Suppose* $f\in L^2[a,b]$ *and* P_n *is a set of all polynomials of degree and smaller n. If there exists a function* $q^*\in P_n$ *such that* $\lim_{n\to\infty} P_n(f)=0$ *, where* $P_n(f)=\inf_{p\in P_n} \|f-p\|_2$ *. Then is called best uniform polynomial approximation to* f *on* $[a,b]$ *.*

Figure 16.

Theorem 2.2 *Let* $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{m,n}^{(\alpha,\beta,M,N)^+}(t)$ *be expanded in terms of generalized Chebyshev wavelets. If* $\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ *is converge, then* (α, β, M, N) *generalized Chebyshev wavelet approximation* $E_{2^k,l}(t)$ *of* f *is* $f_1(t)$.

$$E_{2^k,l}(f) = o((\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}).$$

Proof.

$$\| f - f_1 \|_2^2 = \int_{-1}^{1} | \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t)$$
$$- \sum_{n=0}^{2^k} \sum_{m=0}^{l} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt$$
$$= \int_{-1}^{1} | \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt$$
$$\leq \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt$$
$$\leq \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt$$
$$\leq L \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}.$$

Therefore

$$\| f - f_1(t) \|_2 \leq (\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}},$$

$$E_{2^k,l}(f) = o((\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}).$$

Figure 17.

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Theorem 2.3 *Let* $f(t)=\sum_{n=0}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\Psi_{m,n}^{(\alpha,\beta,M,N)}(t)$ *be expanded in terms of generalized Chebyshev wavelets. If* $\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}|t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ *is converge, then generalized Chebyshev wavelet approximation* $E_{2^k,l}(t)$ *of* f *is* $-f_2(t)$ *and*

$$E_{2^k,l}(f)=o((\sum_{n=0}^{2^k}\sum_{m=l+1}^{\infty}|t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}).$$

Proof.

$$\begin{aligned}\| f-(-f_2)\|_2^2 &= \int_{-1}^{1}|\sum_{n=0}^{2^k}\sum_{m=0}^{l} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t)|^2\omega_{n,m}^{(\alpha,\beta,M,N)}(t)\,dt\\
&+\sum_{n=0}^{2^k}\sum_{m=0}^{l} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t)|^2\omega_{n,m}^{(\alpha,\beta,M,N)}(t)\,dt\\
&=\int_{-1}^{1}|\sum_{n=0}^{2^k}\sum_{m=l+1}^{\infty} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t)|^2\omega_{n,m}^{(\alpha,\beta,M,N)}(t)\,dt\\
&\le\sum_{n=0}^{2^k}\sum_{m=l+1}^{\infty}\int_{-1}^{1}|t_{n,m}|^2|\Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t)|^2\omega_{n,m}^{(\alpha,\beta,M,N)}(t)\,dt\\
&\le\sum_{n=0}^{2^k}\sum_{m=l+1}^{\infty}|t_{n,m}|^2\int_{-1}^{1}|\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2\omega_{n,m}^{(\alpha,\beta,M,N)}(t)\,dt\\
&\le\sum_{n=0}^{2^k}\sum_{m=l+1}^{\infty}|t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}.\end{aligned}$$

Therefore

$$\| f-(-f_2(t))\|_2\le(\sum_{n=0}^{2^k}\sum_{m=l+1}^{\infty}|t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}},$$

$$E_{2^k,l}(f)=o((\sum_{n=0}^{2^k}\sum_{m=l+1}^{\infty}|t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}).$$

Figure 18.

Corollary 2.2 *Let* $f(t)=\sum_{n=0}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\Psi_{m,n}^{(\alpha,\beta,M,N)^+}(t)$ *be expanded in terms of generalized Chebyshev wavelets. If* $\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}|t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ *is converge, then uniform best polynomial approximation of* f *is* $f_1(t)$.

Corollary 2.3 *Let* $f(t)=\sum_{n=0}^{2^k}\sum_{m=0}^{\infty} t_{n,m}\Psi_{m,n}^{(\alpha,\beta,M,N)^-}(t)$ *be expanded in terms of generalized Chebyshev wavelets. If* $\sum_{n=0}^{2^k}\sum_{m=0}^{\infty}|t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ *is converge, then uniform best polynomial approximation of* f *is* $-f_2(t)$.

Figure 19.

Declarations

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