Indonesian Journal of Innovation Studies
Vol. 26 No. 3 (2025): July
Doi: 10.21070/ijins.v26i3.1457

Generalized Jacobi Chebyshev Wavelet Approximation: Perkiraan Wavelet Jacobi Chebyshev Umum

Hamid Mazaheri Tehran Department of Mathematics, Parwan University, Parwan, Afghanastan
Abdulwali Azamsafi Department of Mathematics, Parwan University, Parwan, Afghanastan
Seyed Mohammad Jesmani Department of Mathematics, Parwan University, Parwan, Afghanastan

General Background: Wavelet approximations are fundamental in numerical analysis and signal processing, with classical orthogonal polynomials like Jacobi and Chebyshev serving as key tools due to their strong approximation properties. **Specific Background:** The use of Chebyshev wavelets has been extended through generalized polynomial frameworks, such as Koornwinder's generalization of Jacobi polynomials, offering more flexibility for function approximation on finite intervals. **Knowledge Gap:** Despite existing wavelet frameworks, the integration of generalized Jacobi and Chebyshev structures into a unified wavelet approximation scheme remains underexplored. **Aims:** This study introduces the Generalized Jacobi Chebyshev Wavelet (GJCW) approximation, establishing its theoretical foundations and demonstrating convergence and approximation capabilities. **Results:** It is shown that for a uniformly bounded function expanded in the GJCW basis, the partial sums yield both convergent and best uniform polynomial approximations. **Novelty:** The formulation of a new wavelet approximation based on a hybrid of generalized Jacobi and Chebyshev polynomials constitutes a novel contribution, supported by rigorous recurrence relations and multiresolution analysis. **Implications:** This work enhances the theoretical landscape of wavelet-based function approximation, with potential applications in computational mathematics, signal analysis, and numerical solutions of differential equations.

Highlight :

- Wavelet Construction: The paper defines and constructs *generalized Jacobi Chebyshev wavelets* using orthogonal polynomials.
- Approximation Theory: It proves that if the wavelet series converges, then a *uniform best polynomial approximation* exists.
- Multiresolution Framework: The approach is grounded in *Mallat's multiresolution analysis*, enabling efficient function approximation.

Keywords : Jacobi Polynomials, Chebyshev Wavelets, Multiresolution Analysis, Polynomial Approximation, Orthonormal Basis

INTRODUCTION

Jacobi polynomials $P_n^{\alpha,\beta}(x)$ constitute a category of classical orthogonal polynomials [1]. They are orthogonal about the weight $(1-x)^{\alpha}(1+x)^{\beta}$ on the interval $[-1,1]$. We have $P_n^{(\alpha,\beta)}(x)$ Chebyshev polynomials with $|x| \leq 1$.

Indonesian Journal of Innovation Studies
Vol. 26 No. 3 (2025): July
Doi: 10.21070/ijins.v26i3.1457

n	*Jacobi Polynomial*
0	1
1	$(\alpha+\beta)+\frac{1}{2}(\alpha+\beta+2)x$
2	$-4+\alpha^2-\beta+\beta^2-\alpha(1+2\beta)+2(3\alpha+\alpha^2-\beta(3+\beta))x$ $+\frac{1}{8}(3+\alpha+\beta)(4+\alpha+\beta)x^2$
3	$(\alpha+\beta)(-16+\alpha^2+(-3+\beta)\beta-\alpha(3+2\beta))+3(\alpha+\beta+4)$ $(\alpha^2-(2\beta+1)\alpha+\beta^2-\beta-6)x+3(\alpha+\beta(\alpha+\beta+4)(\alpha+\beta+5)x^2$ $+\frac{1}{48}((\alpha+\beta+4)(\alpha+\beta+5)(\alpha+\beta+6)x^3$

also for $n=4,5,\cdots$

$$P_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\{(1-x)^{\alpha}(1+x)^{\beta}(1-x^2)^n\}.$$

The Jacobi polynomials are generated by the three-term recurrence relation, for scalars $a_n^{(\alpha,\beta)}$, $b_n^{(\alpha,\beta)}$ and $c_n^{(\alpha,\beta)}$,

$$P_{n+1}^{(\alpha,\beta)}(x)=(a_n^{(\alpha,\beta)}x-b_n^{(\alpha,\beta)})P_n^{(\alpha,\beta)}(x)-c_n^{(\alpha,\beta)}P_{n-1}^{(\alpha,\beta)}(x)\ \ n\geq 1,$$

where

$$P_0^{(\alpha,\beta)}(x)=1,\ \ P_1^{(\alpha,\beta)}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta),$$

also

$$a_n^{(\alpha,\beta)}=\frac{(2m+\alpha+\beta+1)(2m+\alpha+\beta)}{2(m+1)(m+\alpha+\beta+1)},$$
$$b_n^{(\alpha,\beta)}=\frac{(\beta^2-\alpha^2)(2m+\alpha+\beta+1)}{2(m+1)(m+\alpha+\beta+1)(m+\alpha+\beta)},$$
$$c_n^{(\alpha,\beta)}=\frac{(2m+\alpha+\beta+2)(m+\alpha)(m+\beta)}{(m+1)(m+\alpha+\beta+1)(2m+\alpha+\beta)}.$$

The Jacobi polynomials $y=P_n^{(\alpha,\beta)}(x)$ solve the linear second-order differential equation [2] [3]

$$(1-x^2)y''+[\beta-\alpha-(\alpha+\beta+2)x]y'+n(n+\alpha+\beta+1)y=0.$$

1. Jacobi Polynomials

Theorem 1. *There are scalars* $a_0^{(\alpha,\beta)},a_1^{(\alpha,\beta)},\cdots,a_n^{(\alpha,\beta)}\in\mathbb{R}$ *such that*

$$P_n^{(\alpha,\beta)}(x)=\sum_{m=0}^{n}a_{n,m}^{(\alpha,\beta)}x^m.$$

Proof. We use mathematical induction [4].
For this

$$P_0^{(\alpha,\beta)}(x)=1,$$
$$P_1^{(\alpha,\beta)}(x)=\frac{1}{2}(\alpha+\beta+2)x+\frac{1}{2}(\alpha-\beta).$$

a. The Hypothesis of Mathematical Induction

Suppose for $0\leq k<n$, we have

$$P_k^{(\alpha,\beta)}(x)=\sum_{m=0}^{k}c_{n,m}^{(\alpha,\beta)}x^m,$$

Therefore

$$P_{n-1}^{(\alpha,\beta)}(x)=\sum_{m=0}^{n-1}c'^{(\alpha,\beta)}_{n,m}x^m$$

and

$$P_{n-2}^{(\alpha,\beta)}(x)=\sum_{m=0}^{n-2}c''^{(\alpha,\beta)}_{n,m}x^m,$$

Indonesian Journal of Innovation Studies
Vol. 26 No. 3 (2025): July
Doi: 10.21070/ijins.v26i3.1457

b. The Rule of Mathematical Induction

For scalars $a_n^{(\alpha,\beta)}, b_n^{(\alpha,\beta)}$ and $c_n^{(\alpha,\beta)}$

$$P_n^{(\alpha,\beta)}(x) = (d_n^{(\alpha,\beta)}x - e_n^{(\alpha,\beta)})P_{n-1}^{(\alpha,\beta)}(x) + f_n^{(\alpha,\beta)}P_{n-2}^{(\alpha,\beta)}(x)$$
$$= (c_0^{(\alpha,\beta)}c'^{(\alpha,\beta)}_{n,0} - f_0^{(\alpha,\beta)}c''^{(\alpha,\beta)}_{n,0}) + \sum_{m=1}^{n-3}[c'_{n,m-1} - c_n^{(\alpha,\beta)}c_{n,m}^{(\alpha,\beta)} - f_n^{(\alpha,\beta)}c"_{n,m}]x^m$$
$$+(c_{n-2}^{(\alpha,\beta)}c'^{(\alpha,\beta)}_{n,n-2} - f_{n-2}^{(\alpha,\beta)}c"^{(\alpha,\beta)}_{n,n-2})x^{n-2} + c_{n-1}^{(\alpha,\beta)}c'^{(\alpha,\beta)}_{n,n-1}x^{n-1} + c'^{(\alpha,\beta)}_{n,n-1}d_n^{(\alpha,\beta)}x^n$$
$$= \sum_{m=0}^{n} h_{n,m}^{(\alpha,\beta)}x^m.$$

Theorem 2.

Let $P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} a_{n,m}^{(\alpha,\beta)}x^m$ be a representation of Jacobi polynomials. Then The following conditions are satisfying: [5] [6]

i) $a_{n,m}^{\alpha,\beta} = 0$ for all $m \geq n$,

ii) $(m-1)(m+2)a_{n,m+2}^{\alpha,\beta} - (\beta-\alpha)(m+1)a_{n.m+1}^{\alpha,\beta} + [-m(m-1) - m(\alpha+\beta+2) + n(n+\alpha+\beta+1)]a_{n,m}^{\alpha,\beta} = 0$ for all $0 \leq m \leq n$.

Proof. Suppose $y = P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} a_{n,m}^{(\alpha,\beta)}x^m$, then

$$y' = \sum_{m=1}^{n} ma_{n,m}^{(\alpha,\beta)}x^{m-1} = \sum_{m=0}^{n-1}(m+1)a_{n,m+1}^{(\alpha,\beta)}x^m$$
$$y'' = \sum_{m=2}^{n} m(m-1)a_{n,m}^{(\alpha,\beta)}x^{m-2} = \sum_{m=0}^{n-2}(m+1)(m+2)a_{n,m+2}^{(\alpha,\beta)}x^m.$$

We put in above equation we have [7] [8]

$$\sum_{m=0}^{n-2}(m+1)(m+2)a_{n,m+2}^{(\alpha,\beta)}x^m - \sum_{m=2}^{n} m(m-1)a_{n,m}^{(\alpha,\beta)}x^m + (\beta-\alpha)\sum_{m=0}^{n-1}(m+1)a_{n,m+1}^{\alpha,\beta}x^m - ((\alpha+\beta$$
$$+2)\sum_{m=0}^{n} ma_{n,m}^{\alpha,\beta}x^m + n(n+\alpha+\beta+1)\sum_{m=0}^{n} a_{n,m}^{\alpha,\beta}x^m = 0,$$

it follows that [9] [10]

$$\sum_{m=0}^{n} -(m-1)(m+2)a_{n,m+2}^{\alpha,\beta} - (\beta-\alpha)(m+1)a_{n.m+1}^{\alpha,\beta}$$
$$+[-m(m-1) - m(\alpha+\beta+2) + n(n+\alpha+\beta+1)]a_{n,m}^{\alpha,\beta}x^m = 0.$$

2. Generalized Jacobi Chebyshev Wavelets

Let $\alpha > -1, \beta > -1, M \geq 0$ and $N \geq 0$. Koornwinder in [11] [12] it is demonstrated that the generalised Jacobi polynomials $\{P_n^{(\alpha,\beta,M,N)}(x)\}_{n=0}^{\infty}$ can be written as

$$P_n^{(\alpha,\beta,M,N)}(x) = P_n^{(\alpha,\beta)}(x) + MQ_n^{(\alpha,\beta)}(x) + NR_n^{(\alpha,\beta)}(x) + MNS_n^{(\alpha,\beta)}(x),\ n = 0,1,2,\ldots,$$

where

$$Q_0^{(\alpha,\beta)}(x) = R_0^{(\alpha,\beta)}(x) = S_0^{(\alpha,\beta)}(x) = 0,$$

and for $n = 1,2,3,\cdots$,

$$Q_n^{(\alpha,\beta)}(x) = \frac{(\beta+2)_{n-1}(\alpha+\beta+2)_{n-1}}{(\alpha+1)_n n!} \times [n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) - (\beta+1)(x-1)DP_n^{(\alpha,\beta)}(x)],$$

$$R_n^{(\alpha,\beta)}(x) = \frac{(\alpha+2)_{n-1}(\alpha+\beta+2)_{n-1}}{(\beta+1)_n n!} \times [n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) - (\alpha+1)(x+1)DP_n^{(\alpha,\beta)}(x)],$$

and

$$S_n^{(\alpha,\beta)} = \frac{(\alpha+\beta+2)_n(\alpha+\beta+2)_{n-1}}{(\alpha+1)(\beta+1)n!(n-1)!} \times [n(n+\alpha+\beta+1)P_n^{(\alpha,\beta)}(x) - \{(\beta+1)(x-1) + (\alpha+1)(x+1)\}DP_n^{(\alpha,\beta)}(x).$$

Since $P_n^{(\alpha,\beta,M,N)}(x), R_n^{(\alpha,\beta)}, Q_n^{(\alpha,\beta)}, S_n^{(\alpha,\beta)}$ are linear combination of $P_n^{(\alpha,\beta)}, DP_n^{(\alpha,\beta)}$, we have [13] [14]

Indonesian Journal of Innovation Studies
Vol. 26 No. 3 (2025): July
Doi: 10.21070/ijins.v26i3.1457

Proposition 1.

The generalized Jacobi polynomials $P_n^{\alpha,\beta,M,N}(x)$ *are generated by the three-term recurrence relation, for scalars* $d_n^{(\alpha,\beta,M,N)}, e_n^{(\alpha,\beta,M,N)}$ *and* $f_n^{(\alpha,\beta,M,N)}$*, for* $n \geq 1$

$$P_{n+1}^{(\alpha,\beta,M,N)}(x) = (d_n^{(\alpha,\beta,M,N)}x - e_n^{(\alpha,\beta,M,N)})P_n^{(\alpha,\beta,M,N)}(x) - f_n^{(\alpha,\beta,M,N)}P_{n-1}^{(\alpha,\beta,M,N)}(x).$$

In the following table, we define generalized Chebyshev wavelets. (see [2], [4], [7], [8], [9], [10], [11], [15]) Suppose $k \in N$ (degree of multiresolution), $m \geq 0, n = 1,2,\cdots,2^k$

$$\Psi_{n,m}^{(\alpha,\beta,M,N)}(t) = \begin{cases} \sqrt{\frac{2^{k+1}}{n}} P_m^{(\alpha,\beta,M,N)}(2^k t - 2n + 1)) & t \in [\frac{n-1}{2^{k-1}}, \frac{n}{2^{k-1}}]. \\ 0 & other9wise \end{cases}$$

A function $f \in L^2[-1,1)$ is expanded by generalized Chebyshev wavelets series as

$$f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} c_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t),$$

where

$$c_{n,m} = \int_{-1}^{1} f(t)\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)\omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt,$$

and $\omega_{n,m}^{(\alpha,\beta,M,N)}$ is the weight function of (α, β, M, N) generaliized Chebyshev polynomials. Also

$$\int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt = L_{n,m}^{(\alpha,\beta,M,N)},$$

It is necessary to study multiresolution analysis and Mallat's Theorem for wavelet approximation. [16]

Definition 1.

a. Multiresolution Analysis

An MRA with scaling function ϕ *constitutes a collection of closed subspaces*

$\{V_j\}_{j\in Z}$ *of* $L^2(\mathbb{R})$*, such that*
(i) $V_j \subset V_{j+1}$;
(ii) $f(x) \in V_j \Leftarrow f(2x) \in V_{j+1}$;
(iii) $\overline{\cup V_j} = L^2(\mathbb{R})$,
(iv) $\cap V_j = 0$;
(v) There exists a function $\phi \in V_0$ such that the collection $\{\phi(x-k): k \in \mathbb{Z}\}$ is a Riesz basis of V_0.

The series of wavelet subspaces W_j of $L^2(\mathbb{R})$ is such that $V_j \perp W_j$, for all$j$ and $V_{j+1} = V_j \oplus W_j$. Closure of $\oplus W_j$ is dense in $L^2(\mathbb{R})$ for L^2 norm.

We now present Mallat's theorem, which ensures that in the context of an orthogonal multiresolution analysis (MRA), an orthonormal basis exists for $L^2(\mathbb{R})$ exists. These basis functions are essential in wavelet theory, facilitating the development of sophisticated computational algorithms.

Lemma 1.

(Mallat's Theorem) In the context of an orthogonal multiresolution analysis (MRA) characterised by a scaling function ϕ, a corresponding wavelet exists $\psi \in L^2(\mathbb{R})$ *such that for each* $j \in \mathbb{Z}$*, the family* $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* W_j*. Hence the family* $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ *is an orthonormal basis for* $L^2(\mathbb{R})$*.*

Indonesian Journal of Innovation Studies
Vol. 26 No. 3 (2025): July
Doi: 10.21070/ijins.v26i3.1457

Definition 2.

(i) Let family $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$ *be an orthonormal basis for* $L^2(\mathbb{R})$ *and* $P_n(f)$ *the orthogonal projection of* $L^2([-1,1])$ *onto* V_n*. Then*

$$P_n(f) = \sum_{-\infty}^{\infty} < f, \psi_{n,k} > \psi_{n,k},\ n = 1,2,3,\cdots$$

(ii) The wavelet approximation of the Chebyshev polynomial is defined by

$$E_n(f) = \| f - P_n(f) \|_2 = \int_{-1}^{1} |f(t) - P_n(f)(t)|^2 dt = o(\phi(n)),\quad o(\phi(n))\ is\ a\ small\ function.$$

Theorem 1.

Let $f \in L^2([-1,1])$ *be a uniformly bounded function and* $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)$ be expanded in terms of generalized Chebyshev wavelets and the series $\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ be convergent. Then generalized Chebyshev wavelet approximation f, for every M is the partial sums

$$u_{2^k,M-1}^{(\alpha,\beta,M,N)}(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{M-1} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t),$$

and

$$E_{2^k,M-1}(f) = o((\sum_{n=0}^{2^k} \sum_{m=M}^{\infty} L_{n,m}^{(\alpha,\beta,M,N)} |t_{n,m}|^2)^{\frac{1}{2}}).$$

Proof. We have

$$\begin{aligned}
&\| f - u_{2^k,M-1^{(\alpha,\beta,M,N)}} \|_2^2 \\
&= \int_{-1}^{1} |\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t) \\
&- \sum_{n=0}^{2^k} \sum_{m=0}^{M-1} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&= \int_{-1}^{1} |\sum_{n=0}^{2^k} \sum_{m=M}^{\infty} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&\leq \sum_{n=0}^{2^k} \sum_{m=M}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&= \sum_{n=0}^{2^k} \sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}
\end{aligned}$$

Therefore $\| f - s_{M-1} \|_2 \leq (\sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}$. That is

$$E_{2^k,M-1}(f) = o((\sum_{n=0}^{2^k} \sum_{m=M}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}),$$

Definition 2.

Suppose $f \in L^2[a,b]$ *and* P_n *is a set of all polynomials of degree n and smaller n. If there exists a function* $q^* \in P_n$ *such that* $lim_{n\to\infty} P_n(f) = 0$*, where* $P_n(f) = inf_{p\in P_n} \| f - p \|_2$*. Then* q^* *is called best uniform polynomial approximation to* f *on* $[a,b]$*.*

Corollary 1.

Let $f \in L^2([-1,1])$ *be a uniformly bounded function and* $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)$ be expanded in terms of generalized Chebyshev wavelets and the series $\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ be convergent. For every $M \geq 0$, the partial sums

$$u_{2^k,M-1}^{(\alpha,\beta,M,N)}(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{M-1} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t).$$

is uniform best polynomial approximation for f.

Suppose $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m}\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)$, we put

$$f_1(x) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t),$$

and

$$f_2(x) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t),$$

then

$$f = f_1 - f_2.$$

Theorem 2.

Let $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{m,n}^{(\alpha,\beta,M,N)^+}(t)$ *be expanded in terms of generalized Chebyshev wavelets. If* $\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ *is converge, then* (α, β, M, N) *generalized Chebyshev wavelet approximation* $E_{2^k,l}(t)$ *of* f *is* $f_1(t)$.

$$E_{2^k,l}(f) = o((\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}).$$

Proof.

$$\begin{aligned}
&\| f - f_1 \|_2^2 = \int_{-1}^{1} | \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t) \\
&- \sum_{n=0}^{2^k} \sum_{m=0}^{l} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&= \int_{-1}^{1} | \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&\leq \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)^+}(t)|^2 \omega_{n.m}^{(\alpha,\beta,M,N)}(t)dt \\
&\leq \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n.m}^{(\alpha,\beta,M,N)}(t)dt \\
&\leq L \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}.
\end{aligned}$$

Therefore

$$\| f - f_1(t) \|_2 \leq (\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}},$$

$$E_{2^k,l}(f) = o((\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}).$$

Theorem 3.

Let $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{m,n}^{(\alpha,\beta,M,N)}(t)$ *be expanded in terms of generalized Chebyshev wavelets. If* $\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ *is converge, then generalized Chebyshev wavelet approximation* $E_{2^k,l}(t)$ *of* f *is* $-f_2(t)$ *and*

$$E_{2^k,l}(f) = o((\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}).$$

Proof.

$$\begin{aligned}
&\| f - (-f_2) \|_2^2 = \int_{-1}^{1} | \sum_{n=0}^{2^k} \sum_{m=0}^{l} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&+ \sum_{n=0}^{2^k} \sum_{m=0}^{l} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&= \int_{-1}^{1} | \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} t_{n,m} \Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&\leq \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} \int_{-1}^{1} |t_{n,m}|^2 |\Psi_{n,m}^{(\alpha,\beta,M,N)^-}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&\leq \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 \int_{-1}^{1} |\Psi_{n,m}^{(\alpha,\beta,M,N)}(t)|^2 \omega_{n,m}^{(\alpha,\beta,M,N)}(t)dt \\
&\leq \sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}.
\end{aligned}$$

Therefore

$$\| f - (-f_2(t)) \|_2 \leq (\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}},$$

$$E_{2^k,l}(f) = o((\sum_{n=0}^{2^k} \sum_{m=l+1}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)})^{\frac{1}{2}}).$$

Corollary 2.

Let $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{m,n}^{(\alpha,\beta,M,N)^+}(t)$ *be expanded in terms of generalized Chebyshev wavelets. If* $\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ *is converge, then uniform best polynomial approximation of* f *is* $f_1(t)$.

Corollary 3.

Let $f(t) = \sum_{n=0}^{2^k} \sum_{m=0}^{\infty} t_{n,m} \Psi_{m,n}^{(\alpha,\beta,M,N)^-}(t)$ *be expanded in terms of generalized Chebyshev wavelets. If* $\sum_{n=0}^{2^k} \sum_{m=0}^{\infty} |t_{n,m}|^2 L_{n,m}^{(\alpha,\beta,M,N)}$ *is converge, then uniform best polynomial approximation of* f *is* $-f_2(t)$.

References

1. S. Altinkaya and S. Yalcin, “The (p,q)-Chebyshev Polynomial Bounds of a General Bi- Univalent Function Class,” Boletín de la Sociedad Matemática Mexicana (3rd Series), vol. 26, no. 2, pp. 341–348, 2020.
2. E. Brandi and P. E. Ricci, “Some Properties of the Pseudo-Chebyshev Polynomials of Half- Integer Degree,” Tbilisi Mathematical Journal, vol. 12, no. 4, pp. 111–121, 2019.
3. M. Cakmak and K. A. Uslu, “A Generalization of Chebyshev Polynomials with Well-Known Kinds and Transition Relations,” Acta Universitatis Apulensis Mathematica-Informatica, no. 57, pp. 19–30, 2019.
4. M. R. Capobianco and W. Themistoclakis, “Interpolating Polynomial Wavelets on [-1,1],” Advances in Computational Mathematics, vol. 23, no. 4, pp. 353–374, 2005.
5. M. Dehghan and M. R. Eslahchi, “Best Uniform Polynomial Approximation of Some Rational Functions,” Computers & Mathematics with Applications, vol. 59, no. 1, pp. 382–390, 2010.
6. M. R. Eslahchi and M. Dehghan, “The Best Uniform Polynomial Approximation to the Class of the Form $1/(a^2 \pm x^2)$,” Nonlinear Analysis: Theory, Methods & Applications, vol. 71, no. 3–4, pp. 740–750, 2009.
7. B. Fischer and J. Prestin, “Wavelets Based on Orthogonal Polynomials,” Mathematics of Computation, vol. 66, no. 220, pp. 1593–1618, 1997.
8. T. Kilgore and J. Prestin, “Polynomial Wavelets on the Interval,” Constructive Approximation, vol. 12, no. 1, pp. 95–110, 1996.
9. L. S. Kumar, S. Mishra, and S. K. Awasthi, “Error Bounds of a Function Related to Generalized Lipschitz Class via the Pseudo-Chebyshev Wavelet and Its Applications in the Approximation of Functions,” Carpathian Mathematical Publications, vol. 14, no. 1, pp. 29–48, 2022.
10. H. K. Nigam, R. N. Mohapatra, and K. Murari, “Wavelet Approximation of a Function Using Chebyshev Wavelets,” Thai Journal of Mathematics, pp. 197–208, 2020.
11. D. Marchi, S. Elefante, G. Marchetti, F. Mariethoz, and J. Zacharie, “More Properties of (β,γ)-Chebyshev Functions and Points,” Journal of Mathematical Analysis and Applications, vol. 528, no. 2, Paper no. 127603, 13 pp., 2023.
12. T. H. Koornwinder, “Orthogonal Polynomials with Weight Function (1–x)^α (1+x)^β + Mδ(x+1) + Nδ(x–1),” Canadian Mathematical Bulletin, vol. 27, no. 2, pp. 205–214, 1984.
13. S. M. Jesmani, H. Mazaheri, and S. Shojaeian, “Wavelet Approximation with Chebyshev,” Iranian Journal of Numerical Analysis and Optimization, vol. 14, no. 1, pp. 315–329, 2024.
14. J. C. Mason and D. C. Handscomb, Chebyshev Polynomials, Boca Raton, FL, USA: Chapman and Hall/CRC, 2003.
15. T. J. Rivlin, An Introduction to the Approximation of Functions, New York, NY, USA: Dover Publications, 1981.
16. M. Tasche, “Polynomial Wavelets on [-1,1],” in Approximation Theory, Wavelets and Applications, NATO ASI Series C: Mathematical and Physical Sciences, vol. 454, Dordrecht, Netherlands: Kluwer Academic Publishers, 1995, pp. 497–512.